Uncertainty Analysis on the Righthand Side for MILP Problems

Zhenya Jia and Marianthi G. Ierapetritou
Dept. of Chemical and Biochemical Engineering, Rutgers University, Piscataway, NJ 08854

DOI 10.1002/aic.10842
Published online April 10, 2006 in Wiley InterScience (www.interscience.wiley.com).

A systematic framework is developed to solve the parametric mixed integer linear programming (pMILP) problems where uncertain parameters are present on the right-hand side (RHS) of the constraints. For the case of multiple uncertain parameters, a new algorithm of multiparametric linear programming (mpLP) is proposed, which solves a number of nonlinear problems (NLP) iteratively. At each iteration, a point at which the objective value cannot be represented by the current optimal functions is found, and the new optimal function is included in the next iteration. Given the range of uncertain parameters in a MILP problem, the output of this proposed framework is a set of optimal integer solutions and their corresponding critical regions and optimal functions. A number of examples are presented to illustrate the applicabilities of the proposed approach and comparison with existing techniques. © 2006 American Institute of Chemical Engineers *AIChE J,* 52: 2486–2495, 2006
Keywords: mixed integer linear programming, parametric, uncertainty

Introduction

A number of problems from the area of process design and operations are commonly formulated as mixed integer linear programming (MILP) problems. One way to incorporate uncertainty into these problems is using MILP sensitivity analysis and parametric programming methods. Using parametric programming to address the issue of uncertainty provides the best available information regarding the solution variability since it identifies the optimal solutions to cover the whole range of uncertainty. As will be shown in example 2 for a scheduling problem a number of alternative schedules are generated that are optimal for different demand realizations. Similarly for the design and synthesis problem a parametric solution of the corresponding optimization problem will provide not only how the objective changes with respect to uncertainty, but also different design and plant choices depending on the uncertainty realization.

The main limitation of most existing methods is that they can only be applied to problems with a single uncertain parameter or several uncertain parameters varying in a single direction. A number of approaches have been developed for parametric integer programming problems that involve a single parameter/scalar variation, basically including implicit enumeration methods (Roodman,[1] Piper and Zoltners[2]), branch and bound methods (Roodman,[3] Marsten and Morin,[4] Ohtake and Nishida[5]), and cutting plane methods (Holm and Klein,[6] Jenkins and Peters[7]), and so on. A detailed literature review can be found in Jenkins.[8]

Jenkins' approach is extended by Crema[9] for the multiparametric 0–1 integer linear programming (ILP) problem considering the perturbation of the constraint matrix, the objective function, and the RHS vector. The proposed algorithm iteratively solves a nonlinear problem, which can be converted to an equivalent MILP formulation, in order to obtain a complete multiparametrical analysis.

Acevedo and Pistikopoulos[10] proposed a parametric programming approach for the analysis of linear process engineering problems under uncertainty. The procedure is based on the solution of multiparametric linear programming (mpLP) at each node of the B&B tree, then compares and identifies the different optimal integer solutions and their corresponding

Correspondence concerning this article should be addressed to M. G. Ierapetritou at marianth@sol.rutgers.edu.

optimal value functions. Pertsinidis et al.[11] developed an algorithm for MILP sensitivity analysis. At each iteration, the LP sensitivity analysis results and a cut that excludes the current integer solution are incorporated to a MILP problem so as to find the breaking point, and the successor optimal integer solution. Their ideas were extended by Dua and Pistikopoulos,[12] by decomposing the mp-MILP into two subproblems, and then iterating between them. The first subproblem is obtained by fixing the integer variables, resulting in a mpLP problem, whereas the second subproblem is obtained by relaxing the parameters as variables, leading to a MILP problem.

The problem of RHS multiparametric linear problem was first addressed by Gal and Nedoma.[13] Their algorithm is based on the Simplex algorithm for deterministic LPs. It starts with an initial optimal basis at a feasible point and moves to each of its possible neighbor bases by one dual step to determine the new optimal solution. This procedure is repeated until there is no optimal basis that still has unexamined neighbors. A geometric approach is proposed by Borrelli et al.,[14] which is based on the direct exploration of the parameter space, and their definition of critical regions is not associated with bases but with the set of active constraints.

In this work, we focus on the parametric MILP problems with RHS uncertain parameters which are allowed to vary independently. The proposed solution procedure starts with the B&B tree of the MILP problem at the nominal values of the uncertain parameters and requires two iterative steps: LP/mpLP sensitivity analysis and updating the B&B tree. Rather than checking all the neighbor bases of the associated LP tableau, a novel algorithm is developed for mpLP problems that can determine the optimal functions, and their corresponding critical regions without constructing the LP tableaus.

This article is organized as follows. The details of the proposed approach for the cases of single and multiple uncertain parameters are presented in the next section. A number of case studies follow to illustrate the steps of the proposed approach including a scheduling problem to demonstrate the importance of the proposed approach to address the problem of uncertainty in process operations. The work is summarized in the last section, and some of the ideas for future developments are discussed.

Proposed Framework

For the general mixed integer problem

$$(\text{P1}) \quad \min \quad z = cx$$

$$\text{subject to} \quad Ax \geq \theta$$

$$x \geq 0, \quad x_j \; integer, \quad j = 1, \ldots, k$$

Assuming a perturbation of problem RHS parameter values such that

$$Ax \geq \theta + \Delta\theta$$

The aim of this work is to investigate the effect of $\Delta\theta$ on the optimal solution x, and objective value z. First the original problem is solved at a starting point following a Branch and Bound solution procedure, and the dual information λ^p, z^p are collected at each leaf node. Then LP sensitivity analysis is performed for the relaxed LP problems at the leaf nodes. The LP sensitivity analysis procedure for multiple uncertain parameters is more complicated than that of the single uncertain parameter, therefore, the detailed steps for these two cases will be presented separately in this section. When only one parameter is involved, the objective varies with the uncertain parameter in one direction. In order to identify the next optimal function, we can slightly increase the value of the uncertain parameter beyond the current optimal basis. The new dual information can then be found so as to determine the next optimal integer solution and update the B&B tree. However, when the objective value is a function of multiple parameters, it could be difficult to find a way to identify all the neighboring bases. Therefore, a new algorithm is developed as will be presented later. After the mpLP sensitivity analysis at the leaf nodes, a comparison procedure is required in order to update the B&B tree. The optimal functions at the leaf nodes are compared with the current upper bound in the same critical region. If an integer leaf node can provide a better objective then the upper bound is updated in that region. If it's not an integer node but has a better objective then it should be branched. The mpLP is performed at the new leaf nodes and the comparison procedure continues. This procedure stops when no further branching is required.

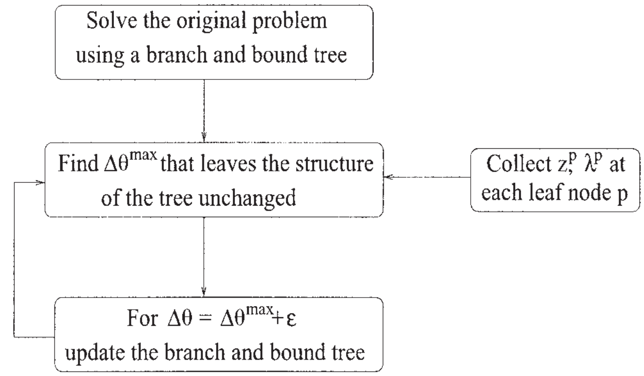

Figure 1. Flow chart of proposed approach for single uncertain parameter case.

Single Uncertain Parameter

For the case of single uncertain parameter, the proposed approach follows the basic ideas of the interactive reference point approach proposed by Alves and Climaco[15] presented for multiple objective MILP problems. The proposed framework is shown in Figure 1.

First, the problem is solved at the nominal values of the uncertain parameters using a branch and bound solution approach, and the dual information λ^p, z^p is collected at each leaf node p. Assuming that the optimal solution is found at node 0, the LP sensitivity analysis is then performed at node 0 to determine the range $\Delta\theta^{basis}$ within which the current optimal basis does not change. We need to find the perturbation $\Delta\theta^{\max}$ such that the structure of the branch and bound remains the same. $\Delta\theta^{\max}$ can be found through the following equation

$$\Delta\theta^{\max} = \min\left\{\Delta\theta^{basis}, \min\left\{\frac{z^p - z^0}{\lambda^0 - \lambda^p}\right\}\right\}$$

Where z^0 and λ^0 are the objective value and dual multiplier at the optimal node 0, respectively. Note that only the positive $(z^p - z^0/\lambda^0 - \lambda^p)$ need to be considered, because the negative one means that node p can never provide a better solution than node 0 at a certain point. Three cases can be observed as follows:

Case 1: $\Delta\theta^{max} = 0$.

Case 2: $\Delta\theta^{max} = \min\{z^p - z^0/\lambda^0 - \lambda^p\}$.

Case 3: $\Delta\theta^{max} = \Delta\theta^{basis}$.

The next step is to update the tree when $\Delta\theta = \Delta\theta^{max} + \varepsilon$ is slightly larger than $\Delta\theta^{max}$. For the three cases above we have:

Case 1: The optimal node yields noninteger solution—continue the branching procedure on node 0 and find the new optimal node, which will be updated as node 0.

Case 2: The optimal node is intersected by other leaf node, which means that $z^0(\Delta\theta^{max}) = z^p(\Delta\theta^{max})$ for some node p. Update node p as the optimal node 0.

Case 3: The optimal node 0 still provides optimal solution, but the basis has changed.

The procedure continues by determining the new $\Delta\theta^{max}$, and these steps are executed repeatedly until no feasible solution exists beyond the current range.

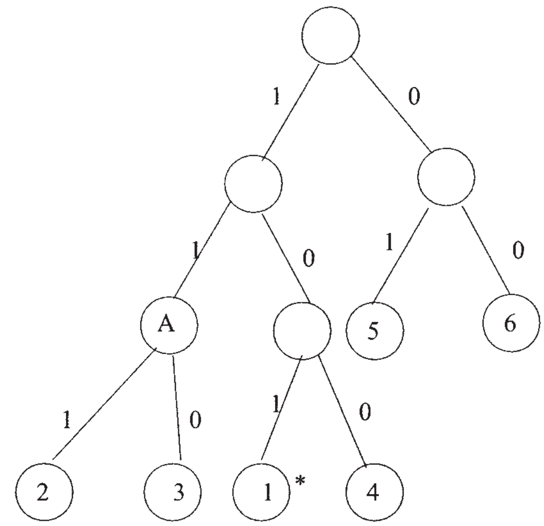

Figure 3. Branch and bound tree for the original problem.

Multiple Uncertain Parameters

This subsection presents the detailed steps (Figure 2) of the proposed approach to deal with the case of multiple uncertain parameters. Assuming for simplicity in the presentation that we want to investigate two parameters, θ_a and θ_b, changing in the range of $[a_0, a_0 + \Delta a]$ and $[b_0, b_0 + \Delta b]$. The MILP problem is first solved at (a_0, b_0) using branch and bound algorithm, and the optimal solution is found at node 1 (Figure 3). Other leaf nodes of the B&B tree are denoted as node 2, node 3, . . . , node n. Note that only the information at the leaf nodes are required. If (a_0, b_0) becomes (a', b'), and there exists a new optimal solution, it can always be uncovered by checking or continuing the branching procedure on the current leaf nodes. Assuming that the new optimal solution can be provided by a nonleaf node (node A). With the original data, the relaxed LP problem of node A must have a partial integer solution, otherwise it is a leaf node. With the perturbed data, the LP problem of node A gives the optimal integer solution. According to our proposed method, the next step is to examine the current leaf nodes, which include the subsequent nodes of node A (node 2 and 3). Apparently, if node A yields an integer solution, either node 2 or 3 should provide that solution too.

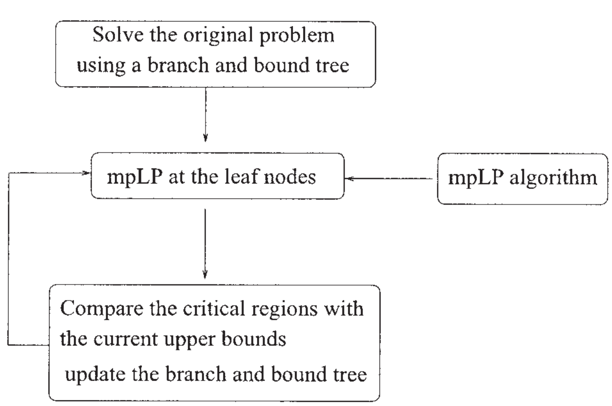

Figure 2. Flow chart of proposed approach for multiple uncertain parameters case.

Then the multiparametric linear programming is performed at each of the leaf nodes including node 1, so as to identify the optimal value functions and their corresponding critical regions in the region of $[a_0, a_0 + \Delta a]$ and $[b_0, b_0 + \Delta b]$. In this work, a new algorithm is proposed for the solution of mpLP. When the mpLP procedure is completed, the output will be a set of optimal functions $z = z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b$, $k = 1, \ldots, K$, where K is the number of critical regions. For any point (θ_a, θ_b) in the range of $[a_0, a_0 + \Delta a]$ and $[b_0, b_0 + \Delta b]$, the objective value cx^* of the relaxed LP problem of that node can be expressed by $\max\{z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b, k = 1, \ldots, K\}$. If the procedure is not complete, then there must exist a point (θ_a, θ_b), such that $\max\{z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b, k = 1, \ldots, K\}$ is less than cx^*. Thus, a bilevel programming problem is formulated as follows

$$\text{(P2)} \quad \max\{\min cx | Ax \geq \theta\} - z$$

$$\text{subject to} \quad z \geq z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b, \quad k = 1, \ldots, K$$

$$a_0 \leq \theta_a \leq a_0 + \Delta a$$

$$b_0 \leq \theta_b \leq b_0 + \Delta b$$

It is proved that linear bilevel programming problems (BLPP) are NP-hard.[16] In order to avoid solving a BLPP, we propose to first convert the relaxed LP problems at the leaf nodes to its dual form, so that the uncertain parameters appear in the objective function.

$$\text{(P3)} \quad \max \quad \theta y$$

$$\text{subject to} \quad A^T y \leq c$$

$$a_0 \leq \theta_a \leq a_0 + \Delta a$$

$$b_0 \leq \theta_b \leq b_0 + \Delta b$$

$$y \geq 0$$

Replacing the inside optimization problem of (P2) by its dual (P3), the problem can be solved as a single optimization problem (P4).

Starting from the nominal point (a_0, b_0), an optimal function is obtained with respect to the uncertain parameters as follows

$$z = z^{(1)} + \lambda^{(1)}\theta_a + \beta^{(1)}\theta_b$$

and incorporated in problem (P4). Note that the problem is nonlinear due to the bilinear term in the objective function. In problem (P4), the objective function is to maximize the gap between the optimal objective value θy at any point (a, b) in the uncertain range and the maximum value of the optimal function, which is $\max(z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b)$, and, therefore, takes the form $(\theta y - z)$. The constraints contain the original constraints and the current optimal functions $z = z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b$, hence, all the constraints of problem (P4) are linear. If the objective value of the (P4) model is nonzero, it means that there exists at least one point (θ_a, θ_b) at which its real objective value cannot be represented by any of the current objective value functions. Therefore, the objective value function at that point is $z = z' + \lambda'\theta_a + \beta'\theta_b$ and should be included in the next iterations. At each iteration, (P4) problem is solved in order to identify any uncovered region. This procedure terminates when the objective value for problem (P4) is 0, which means that the entire uncertain parameter range is covered by the existing objective value functions

$$\text{(P4)} \quad \max \quad \theta y - z$$

$$\text{subject to} \quad A^T y \leq c$$

$$z \geq z^{(k)} + \lambda^{(k)}\theta_a + \beta^{(k)}\theta_b, \quad k = 1, \ldots, K$$

$$a_0 \leq \theta_a \leq a_0 + \Delta a$$

$$b_0 \leq \theta_b \leq b_0 + \Delta b$$

$$y \leq 0$$

Since (P4) is a nonconvex problem, it is solved using a global optimization solver GAMS/BARON,[17] which relies on *branch-and-reduce* algorithm.

Therefore, by performing mpLP at each leaf node p, a number of critical regions (K), $CR_p^{(1)}, CR_p^{(2)}, \ldots, CR_p^{(K)}$ are identified and in each $CR_p^{(k)}$, $k = 1, \ldots, K$, the optimal value $z_p^{*(k)}$ is expressed as $z_p^{*(k)} = z_p^{(k)} + \lambda_p^{(k)}\theta_a + \beta_p^{(k)}\theta_b$.

Similar to the procedures presented for the single uncertain parameter case, the next step is to update the B&B tree. However, this step becomes more complicated since the parameters vary in multiple directions. The main procedure is to compare the critical regions of the leaf nodes with the current upper bounds and finally identify a set of new critical regions, and their corresponding objective function values and optimal integer solutions. At the beginning, the upper bounds CR^{UB} are set to be the critical regions of the current optimal node (node 1), which are $CR_1^{(1)}, CR_1^{(2)}, \ldots, CR_1^{(K)}$. Assuming that we want to compare critical regions CR_1^{UB} and $CR_2^{(2)}$, which have intersection CR^{int}, the following constraint is defined:

$$z_1^{UB} \geq z_2^{*(2)} \tag{1}$$

and a redundancy test for this constraint is solved in CR^{int} as follows:[10]

$$\text{(P5)} \quad \min \quad \varepsilon$$

$$\text{subject to} \quad z_1^{UB} = z_2^{*(2)} + \varepsilon$$

$$z_1^{UB} = z_1^{UB} + \lambda_1^{UB}\theta_a + \beta_1^{UB}\theta_b$$

$$z_2^{(2)} = z_2^{(2)} + \lambda_2^{(2)}\theta_a + \beta_2^{(2)}\theta_b$$

$$\theta_a, \theta_b \in CR^{int}$$

There could be three cases which are listed as follows:

Case 1: Problem (P5) is infeasible. It implies that z_1^{UB} is smaller in CR^{int} and since its a minimization problem, the solution of CR_1^{UB} and $z_1^{UB} = z_1^{UB} + \lambda_1^{UB}\theta_a + \beta_1^{UB}\theta_b$ remains to be the upper bound for CR^{int}, and node 2 will not provide a better integer solution in CR^{int} and, thus, fanthomed for CR^{int}.

Case 2: The solution of problem (P5) is positive and therefore constraint (1) is redundant. This means that z_1^{UB} is larger and, therefore, the solution of $CR_2^{(2)}$ and $z_2^{*(2)} = z_2^{(2)} + \lambda_2^{(2)}\theta_a + \beta_2^{(2)}\theta_b$ are stored for CR^{int}. If node 2 represents an integer solution, then the optimal function for CR^{int} is updated to be $z_2^{*(2)}$. If node 2 doesn't correspond to an integer solution, then the information of this node should be stored and continue the branching procedure on node 2 in CR^{int} to identify new nodes.

Case 3: The solution of problem (P5) is negative which means that constraint (1) is not redundant and, thus, CR^{int} is divided into two parts by constraint $z_1^{UB} \geq z_2^{*(2)}$, z_1^{UB} provides a better solution on one side in which the current upper bound doesn't change, whereas $z_2^{*(2)}$ gives a better solution on the other side, in which $z_2^{*(2)} = z_2^{(2)} + \lambda_2^{(2)}\theta_a + \beta_2^{(2)}\theta_b$, should be stored if node 2 is an integer node, otherwise node 2 will be branched to identify new nodes.

At each iteration, the new leaf nodes in the updated B&B tree will be compared to the current upper bounds, so as to determine the new optimal functions in their intersected region. This procedure stops when no further branching is required, and the uncertainty analysis of the entire uncertain space can be presented by a number of critical regions that contain their corresponding optimal functions and integer solutions.

Comparing to the existing approach,[10] the proposed method solves the mpLP at only the leaf nodes in the B&B tree instead of every node during the branch and bound procedure, and, consequently, reduces the computational efforts significantly as will be shown in the next section through a number of example problems. Moreover, the new mpLP approach can efficiently determine the optimal functions and critical regions without retrieving the LP tableaus and visiting the neighbor bases.

Case Studies

In this section, a number of examples are presented for the cases of single and multiple uncertain parameters so as to better explain the proposed methodologies.

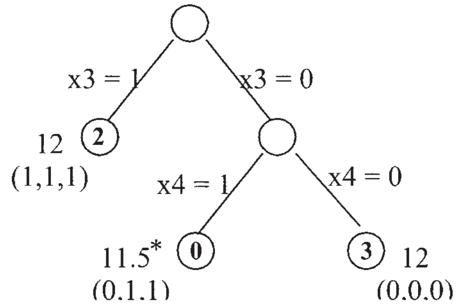

Figure 4. Branch and bound tree at step 1 for example 1.

Examples for Single Uncertain Parameter Case

Example 1

$$\min \quad z = 2x_1 + 3x_2 + 1.5x_3 + 2x_4 + 0.5x_5$$

$$\text{subject to} \quad 2x_1 + x_2 + x_3 \geq 7 \tag{2}$$

$$2x_2 + x_4 + x_5 \geq 4 \tag{3}$$

$$x_3 + x_4 - x_5 \geq 0 \tag{4}$$

$$2x_1 - x_2 - x_3 + x_5 \geq 4 \tag{5}$$

$$1 \leq x \leq 3, \quad x_j \in (0, 1), \quad j = 3, 4, 5$$

For the above MILP problem, we are interested in studying how the objective value changes with respect to the righthand side of the first constraint

$$2x_1 + x_2 + x_3 \geq 7 + \Delta\theta$$

Step 1: Solve the original problem with a B&B tree as shown in Figure 4. The optimal solution is provided by node 0 with $z = 11.5$ and $(x_3, x_4, x_5) = (0, 1, 1)$:

Step 2: Performing linear sensitivity analysis on node 0 gives $\Delta\theta^{\max} = 0$.

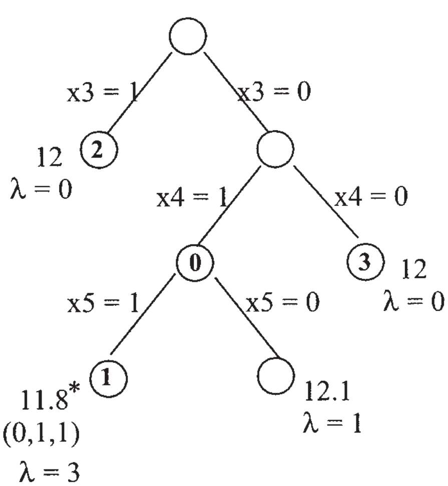

Figure 5. Updated branch and bound tree for example 1.

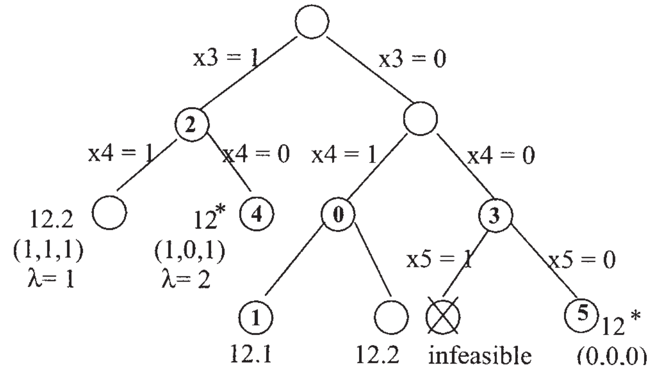

Figure 6. Updated branch and bound tree for example 1.

Step 3: For $\Delta\theta = \Delta\theta^{\max} + \varepsilon = 0.1$ (let $\varepsilon = 0.1$), node 0 yields noninteger solution (Case 1). The updated tree is shown in Figure 5, where the value of dual multiplier λ is associated with each leaf node.

Step 2: Since $\min\{(12.1 - 11.8/3 - 1), (12 - 11.8/3 - 0), (12 - 11.8/3 - 0)\} = 0.2/3$ is within the range [0, 2] obtained from sensitivity analysis, $\Delta a^{\max}$ is equal to 0.2/3 and is given by node 2 and 3.

Step 3: For $\Delta\theta = \Delta\theta^{\max} + \varepsilon = 0.2/3 + 0.1/3 = 0.1$ ($\varepsilon = 0.1/3$), node 1 is intersected by leaf nodes 2 and 3 (Case 2). Therefore, the new optimal solution can be found by continuing the branch and bound procedure on these two nodes, as indicated in Figure 6.

After two more iterations, we found that the problem becomes infeasible when $\Delta\theta$ is greater than 2. Figure 7 presents how the objective value and integer solution change with respect to $\Delta\theta$.

Note that the value of ε is usually chosen to be small enough, such as <2% of θ_0, so that the new $\theta_0 + \Delta\theta$ doesn't exceed the next optimal basis.

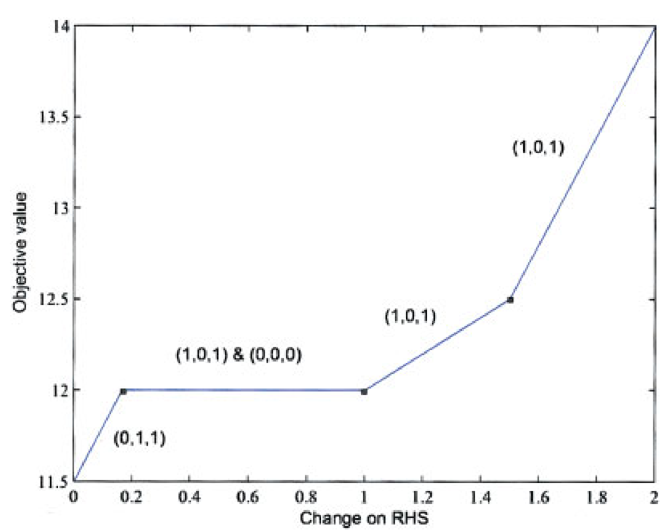

Figure 7. Correlation between $\Delta\theta$ and objective value.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley. com.]

Figure 8. State-task network representation for Example 2.

Example 2

Our second example is a scheduling problem that involves a single product production line consisting of a mixer, a reactor and a purifier as shown in Figure 8. The data for this example are obtained from Ierapetritou and Floudas.[18] The product demand is considered to be the uncertain parameter with the objective function of minimizing production makespan.

In the first step, the problem is solved at the nominal demand value (50). A branch and bound tree is constructed as illustrated in Figure 9, where the value next to each node indicates the objective value of relaxed LP and the binary variables that are branched at each level are shown at the right side. The optimal schedule is found to be schedule 1 with makespan 9.83 h, and schedule 4 and 5 are feasible schedules that have makespan 10.83 h. Note that nodes a, b, c, and d practically all represent the same schedule as node 1, which means they are equivalent schedules.

Step 2: Performing linear sensitivity analysis on node 1, we get $\Delta\theta^{\max} = 0$.

Step 3: For $\Delta\theta = \Delta\theta^{\max} + \varepsilon = 1$ ($\varepsilon = 1$), node 1 yields noninteger solution (Case 1). The updated tree is shown in Figure 10, where schedule 2 provides the optimal schedule with objective value 10.82 h and schedule 3 is a feasible schedule.

Step 2: Sensitivity analysis on node 2 gives range of [0, 10] for the current optimal basis. As illustrated in Figure 10, the dual multipliers at node 3, 4 and 5 are all greater than the one at node 2, which means node 2 will not be intersected by other nodes. Hence, $\Delta\theta^{\max}$ is equal to 10.

Step 3: For $\Delta\theta = \Delta\theta^{\max} + \varepsilon = 10 + 1 = 11$ ($\varepsilon = 1$), node 2 still provides optimal schedule but the basis changes (Case 3). The new optimal makespan is 11.41 h when the demand becomes 62.

Schedule 2 continues to be the optimal schedule in the next iteration, in which $\Delta\theta^{\max} = 40$. After that, the problem becomes infeasible.

After the schedules are determined to cover the demand uncertainty, the schedules are evaluated based on their robustness and expected performance in the face of uncertainty. In this work, corrected SD robustness metric[19] is used to consider 5 scenarios in the demand interval [20, 100]. The average makespan, standard deviation and the nominal makespan are

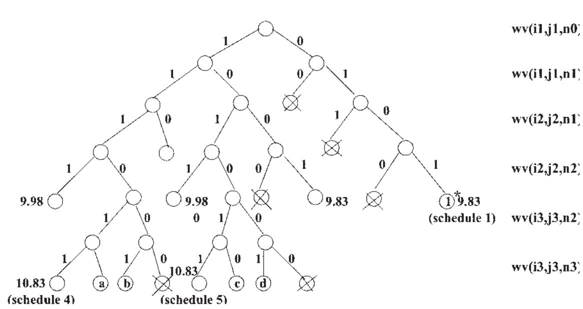

Figure 9. Branch and bound tree at step 1 for example 2.

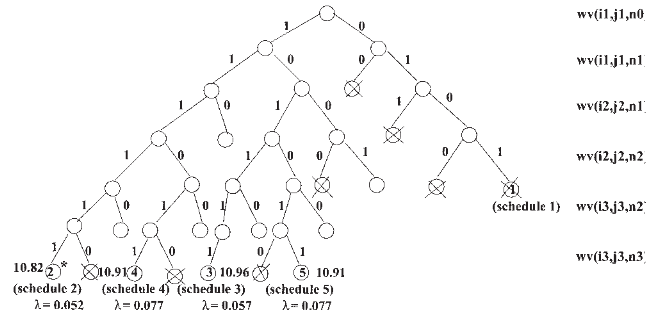

Figure 10. Updated branch and bound tree for example 2.

shown in Table 1. Note that although schedule 1 provides the best solution at nominal point, it has poor performance over the entire interval. Schedule 2 apparently has the highest robustness and provides a short average makespan. Schedule 3 provides a compromise solution with robustness between the ones of schedules 1 and 2 and average and nominal performance close to the best ones given by schedule 2. Schedule 4(5) gives the shortest average makespan but has the poorest robustness.

Examples for Multiple Uncertain Parameters Case

The two examples in Acevedo and Pistikopoulos's work[10] are used for the case of multiple uncertain parameters.

Example 3

$$\min \quad z = -3x_1 - 8x_2 + 4y_1 + 2y_2$$

$$\text{subject to} \quad x_1 + x_2 \le 13 + \theta_1 \tag{6}$$

$$5x_1 - 4x_2 \le 20 \tag{7}$$

$$-8x_1 + 22x_2 \le 121 + \theta_2 \tag{8}$$

$$4x_1 + x_2 \ge 8 \tag{9}$$

$$x_1 - 10y_1 \le 0 \tag{10}$$

$$x_2 - 15y_3 \le 0 \tag{11}$$

$$x \ge 0; \quad y \in \{0, 1\}; \quad 0 \le \theta_1, \theta_2 \le 10$$

This problem contains two continuous variables x_1, x_2 and two binary variables y_1, y_2. θ_1 and θ_2 in the RHS of constraints (Eq. 6) and (Eq. 8) are uncertain parameters varying in the range [0, 10].

Table 1. Comparison of Alternative Schedules for Example 2

	Schedule 1	Schedule 2	Schedule 3	Schedule 4 & 5
H_{nom} (h)	9.83	10.77	10.91	10.83
H_{avg} (h)	14.20	11.56	11.79	11.40
SD_{corr}	5.52	1.61	2.17	6.92

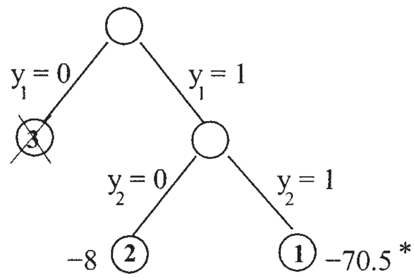

Figure 11. Branch and bound tree for example 3.

Step 1: Solve the original problem

The problem is first solved with B&B solution algorithm for $\theta^0 = (0, 0)$. As shown in Figure 11, both nodes 1 and 2 provide integer solutions, which are $(y_1, y_2) = (1, 1)$ and $(y_1, y_2) = (1, 0)$, respectively, and the optimal values at these two nodes are -70.5 and -8, respectively. Therefore, node 1 gives the optimal solution. Node 3 is found to be an infeasible node and, thus, fanthomed.

Step 2: mpLP on the leaf nodes

Multiparametric linear programming is performed on these two leaf nodes, so as to determine how the objective value z changes with respect to θ_1 and θ_2, and the critical regions in which the optimal functions are valid.

Node 1: The dual multipliers λ_1 and β_1 of the relaxed LP at node 1 are -4.3333 and -0.1667, thus, the optimal function at θ^0 is $z_1^{(1)} = -70.5 - 4.3333\theta_1 - 0.1667\theta_2$. In order to find the critical region in which this function holds, problem (P4) is formulated and solved as follows

$$\max \quad (-13 - \theta_1)d_1 - 20d_2 + (-121 - \theta_2)d_3 + 8d_4 - d_7 - d_8 - z$$

$$\text{subject to} \quad -d_1 - 5d_2 + 8d_3 + 4d_4 - d_5 \le -3 \tag{12}$$

$$-d_1 + 4d_2 - 22d_3 + d_4 - d_6 \le -8 \tag{13}$$

$$10d_5 - d_7 \le 4 \tag{14}$$

$$15d_6 - d_8 \le 2 \tag{15}$$

$$z \ge -70.5 - 4.3333\theta_1 - 0.1667\theta_2 \tag{16}$$

$$0 \le \theta_1, \theta_2 \le 10$$

The solution of this problem identifies if there exists a point (θ_1, θ_2), at which the objective value z^* of the relaxed LP problem cannot be expressed by the current optimal functions. The solution is found to be $(\theta_1, \theta_2) = (10, 0)$ and the objective value is 16.74. The problem is solved using GAMS/BARON within 1 iteration in 0.02 CPU s. The same solution is obtained using GAMS/CONOPT in six iterations, while only a local optimal solution is found with an objective function of 0 using GAMS/MINOS. At $(\theta_1, \theta_2) = (10, 0)$, the optimal function of the LP problem at node 1 is $z_1^{(2)} = -97.0909 - 0.3636\theta_2$. Then the constraint $z \ge -97.0909 - 0.3636\theta_2$ is included in the earlier problem, and the new objective value becomes 0. Therefore, the entire uncertain space can be covered by these two optimal functions.

The critical regions of the two optimal functions can be derived from the inequality $z_1^{(1)} \ge z_1^{(2)} \Rightarrow 0.07333\theta_1 - 0.00333\theta_2 \le 0.45$.

Thus, we have

$$z_1^{(1)} = -70.5 - 4.3333\theta_1 - 0.1667\theta_2$$

$$CR_1^{(1)} = \begin{cases} 0.07333\theta_1 - 0.00333\theta_2 \le 0.45 \\ \theta_2 \le 10 \end{cases}$$

$$z_1^{(2)} = -97.0909 - 0.3636\theta_2$$

$$CR_1^{(2)} = \begin{cases} 0.07333\theta_1 - 0.00333\theta_2 \ge 0.45 \\ \theta_1, \theta_2 \le 10 \end{cases}$$

Node 2: The current optimal function is $z_2 = -8$, which means the objective value doesn't vary with respect to θ_1 and θ_2.

Similarly to node 1, the following problem is formulated and solved and the objective value is found to be 0.

$$\max \quad (-13 - \theta_1)d_1 - 20d_2 + (-121 - \theta_2)d_3 + 8d_4 - d_7 - d_8 - z$$

$$\text{subject to} \quad -d_1 - 5d_2 + 8d_3 + 4d_4 - d_5 \le -3 \tag{17}$$

$$-d_1 + 4d_2 - 22d_3 + d_4 - d_6 \le -8 \tag{18}$$

$$10d_5 - d_7 \le 4 \tag{19}$$

$$15d_6 - d_8 \le 2 \tag{20}$$

$$z \ge 8 \tag{21}$$

$$0 \le \theta_1, \theta_2 \le 10$$

Therefore, the mpLP procedure stops and the result at node 2 is

$$z_2 = 8 \quad CR_2 = \begin{cases} \theta_1 \le 10 \\ \theta_2 \le 10 \end{cases}$$

Step 3: Compare critical regions and determine optimal functions

$CR_1^{(1)}$ and CR_2: Since CR_2 contains the entire uncertainty space, $CR_1^{(1)} \cap CR_2 = CR_1^{(1)}$. Then $z_1^{(1)}$ and z_2 are compared in $CR_1^{(1)}$, which can be achieved by solving the following redundancy test formulation

$$\min \quad \varepsilon$$

$$-70.5 - 4.3333\theta_1 - 0.1667\theta_2 + \varepsilon = -8$$

$$0.07333\theta_1 - 0.00333\theta_2 \le 0.45$$

$$\theta_2 \le 10$$

It is found that $\varepsilon > 0$, therefore $z_1^{(1)} \le z_2$.

$CR_1^{(2)}$ and CR_2: It is trivial that $CR_1^{(2)} \cap CR_2 = CR_1^{(2)}$. The following redundancy test shows that $\varepsilon > 0$, hence constraint $z_1^{(2)} \le z_2$ is redundant in $CR_1^{(2)}$.

$$\min \quad \varepsilon$$

$$-97.0909 - 0.3636\theta_2 + \varepsilon = -8$$

$$0.07333\theta_1 - 0.00333\theta_2 \ge 0.45$$

$$\theta_1,\ \theta_2 \le 10$$

Thus, the final optimal solution for this problem is

$$y^* = (1, 1)$$

$$z_1(\theta) = -70.5 - 4.3333\theta_1 - 0.1667\theta_2$$

$$CR_1 = \begin{cases} 0.07333\theta_1 - 0.00333\theta_2 \le 0.45 \\ \theta_2 \le 10 \end{cases}$$

$$z_2(\theta) = -97.0909 - 0.3636\theta_2$$

$$CR_2 = \begin{cases} 0.07333\theta_1 - 0.00333\theta_2 \ge 0.45 \\ \theta_1,\ \theta_2 \le 10 \end{cases}$$

Compared with the approaches that exist in the literature Acevedo and Pistikopoulos[10] report only the solution z_1 at CR_1, whereas Dua and Pistikopoulos[12] report the complete solution using 1 mpLP and 3 MILPs. The approach we propose in this article uses 1 mpLP and 2 NLPs to obtain the complete parametric solution.

Example 4

The next example involves two continuous variables x_1, x_2, two binary variables y_1, y_2, and three uncertain parameters θ_1, θ_2 and θ_3 that vary within the range of [0, 5]

$$\min \quad z = -3x_1 - 2x_2 + 10y_1 + 5y_2$$

$$\text{subject to} \quad x_1 \le 10 + \theta_1 + 2\theta_2 \tag{22}$$

$$x_2 \le 10 - \theta_1 + \theta_2 \tag{23}$$

$$x_1 + x_2 \le 20 - \theta_2 \tag{24}$$

$$x_1 + 2x_2 \le 12 + \theta_1 - \theta_3 \tag{25}$$

$$x_1 - 20y_1 \le 0 \tag{26}$$

$$x_2 - 20y_2 \le 0 \tag{27}$$

$$-x_1 + x_2 \ge 4 - \theta_3 \tag{28}$$

$$y_1 + y_2 \ge 1 \tag{29}$$

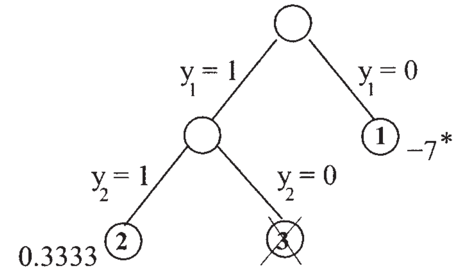

Figure 12. Branch and bound tree for example 4.

$$x \ge 0; \quad y \in \{0, 1\}; \quad 0 \le \theta_1,\ \theta_2,\ \theta_3 \le 5$$

Step 1: Solve the original problem

At $\theta^0 = (0, 0, 0)$, the B&B tree that solves the problem finds two integer solutions $(y_1, y_2) = (0, 1)$, and $(y_1, y_2) = (1, 1)$ at node 1 and node 2, respectively. As illustrated in Figure 12, the optimal value is provided by node 1, which is -7. Node 3 is an infeasible node and, therefore, is fathomed.

Step 2: mpLP on the leaf nodes

Node 1: The optimal function at θ^0 is $z_1^{(1)} = -7 - \theta_1 + \theta_3$, which is then included in the dual of the LP problem so as to identify other optimal functions

$$\max \quad (-10 - \theta_1 - 2\theta_2)d_1 + (-10 + \theta_1 - \theta_2)d_2 + (-20 + \theta_2)d_3 + (-12 - \theta_1 + \theta_3)d_4 + (4 - \theta_3)d_7 + d_8 - d_9 - d_{10} - z$$

$$\text{subject to} \quad -d_1 - d_3 - d_4 - d_5 - d_7 \le -3$$

$$-d_2 - d_3 - 2d_4 + d_7 \le -2$$

$$20d_5 + d_8 - d_9 \le 10$$

$$20d_6 + d_8 - d_{10} \le 5$$

$$z \ge -7 - \theta_1 + \theta_3$$

$$0 \le \theta_1,\ \theta_2,\ \theta_3 \le 5$$

It is found that at $\theta = (5, 0, 0)$, the gap between the LP objective and the value given by the current optimal function $z_1^{(1)}$ is 7. Using GAMS/BARON as the NLP solver, it requires only 1 iteration and 0.02 CPU s. A local optimal solution with objective value 0 is obtained using CONOPT and MINOS. Therefore, the optimal function at $\theta = (5, 0, 0)$, which is $z_1^{(2)} = -15 + 2\theta_1 - 2\theta_2$, is added to the above problem. The mpLP procedure terminates after this iteration and the results at node 1 are

$$z_1^{(1)} = -7 - \theta_1 + \theta_3$$

$$CR_1^{(1)} = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \le 8 \\ \theta_1,\ \theta_2,\ \theta_3 \le 5 \end{cases}$$

$$z_1^{(2)} = -15 + 2\theta_1 - 2\theta_2$$

$$CR_1^{(2)} = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \geq 8 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

Node 2: Starting from θ^0, two additional problems are solved to complete the mpLP. The optimal functions and their corresponding critical regions are as follows:

$$z_2^{(1)} = -0.3333 - 1.6667\theta_1 + 0.3333\theta_3$$

$$CR_2^{(1)} = \begin{cases} 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

$$z_2^{(2)} = -23 + 5\theta_1 - 5\theta_2 - 3\theta_3$$

$$CR_2^{(2)} = \begin{cases} 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \geq 23.3333 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

Step 3: Compare critical regions and determine optimal functions

$CR_1^{(1)}$ and $CR_2^{(1)}$: The intersection of $CR_1^{(1)}$ and $CR_2^{(1)}$ can be determined by combining all the constraints of these two regions

$$\begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \leq 8 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

Then the redundancy test is performed on each of the constraints and $6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333$ is found to be redundant, therefore, the intersection is

$$CR_a^{int} = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \leq 8 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

Another redundancy test shows constraint $z_1^{(1)} \leq z_2^{(1)}$ is redundant in CR_a^{int}, hence, $z_1^{(1)}$ represents the optimal function in CR_a^{int}.

$CR_1^{(1)}$ and $CR_2^{(2)}$: Since $CR_1^{(1)} \cap CR_2^{(1)} = CR_1^{(1)}$, it is trivial that $CR_1^{(1)} \cap CR_2^{(2)} = \varnothing$.

$CR_1^{(2)}$ and $CR_2^{(1)}$: $CR_1^{(2)} \cap CR_2^{(1)} = CR_b^{int}$, and constraint $z_1^{(2)} \leq z_2^{(1)}$ is not redundant

$$CR_b^{int} = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \geq 8 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333 \\ \theta_1 \leq 5, \quad \theta_3 \leq 5 \end{cases}$$

Thus, CR_b^{int} is divided into two uncertain spaces by constraint $z_1^{(2)} \leq z_2^{(1)}$.

$CR_1^{(2)}$ and $CR_2^{(2)}$: $CR_1^{(2)} \cap CR_2^{(2)} = CR_c^{int}$, and constraint $z_1^{(2)} \leq z_2^{(2)}$ is not redundant.

$$CR_c^{int} = \begin{cases} 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \geq 23.3333 \\ \theta_1 \leq 5 \end{cases}$$

Similarly, CR_c^{int} is divided into two uncertain spaces by constraint $z_1^{(2)} \leq z_2^{(2)}$.

To summarize, the final results of MILP parametric analysis for this example are

Optimal integer solution: $y^* = (0, 1)$

$$z_1(\theta) = -7 - \theta_1 + \theta_3$$

$$CR_1 = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \leq 8 \\ \theta_1,\ \theta_2,\ \theta_3 \leq 5 \end{cases}$$

$$z_2 = -23 + 5\theta_1 - 5\theta_2 - 3\theta_3$$

$$CR_{2a} = \begin{cases} 3.6667\theta_1 - 2\theta_2 - 0.3333\theta_3 \leq 15.3333 \\ 3\theta_1 - 2\theta_2 - \theta_3 \geq 8 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333 \\ \theta_1 \leq 5, \quad \theta_3 \leq 5 \end{cases}$$

$$CR_{2b} = \begin{cases} 3\theta_1 - 2\theta_2 - \theta_3 \geq 8 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \geq 23.3333 \\ \theta_1 \leq 5 \end{cases}$$

Optimal integer solution: $y^* = (1, 1)$

$$z_3 = -0.3333 - 1.6667\theta_1 + 0.3333\theta_3$$

$$CR_3 = \begin{cases} 3.6667\theta_1 - 2\theta_2 - 0.3333\theta_3 \geq 15.3333 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \leq 23.3333 \\ \theta_1 \leq 5, \quad \theta_3 \leq 5 \end{cases}$$

$$z_4 = -23 + 5\theta_1 - 5\theta_2 - 3\theta_3$$

$$CR_4 = \begin{cases} 3\theta_1 - 3\theta_2 - 3\theta_3 \leq 8 \\ 6.6667\theta_1 - 5\theta_2 - 3.3333\theta_3 \geq 23.3333 \\ \theta_1 \leq 5 \end{cases}$$

The results obtained are the same as reported in the literature. Our proposed approach requires the solution of 2 mpLPs at two leaf nodes that solve 4 NLP problems, and 3 comparisons of the critical regions. Using the method proposed by Acevedo and Pistikopoulos,[10] it requires a total of 7 mpLPs solved at 5 nodes during the B&B procedure, and 8 comparisons of the critical regions, whereas Dua and Pistikopoulos approach[12] requires the solution of 2 mpLPs and 7 MILPs to identify the parametric solution.

Summary and Future Work

The issue of uncertainty analysis for MILP problems is addressed in this paper. An integrated framework is developed that allows the parameters in the RHS of the MILP formulation to vary independently. It mainly consists of two steps: LP/mpLP sensitivity analysis and updating the B&B tree. For the case of mpLP, a novel algorithm is proposed which solves a set of NLP problems iteratively using the commercially available global optimization solver BARON. It should be noted that due to the nature of nonconvexities (bilinear terms) BARON works extremely well and converges to the global optimal solution in only one iteration in all the examples considered using the default tolerances. The approach is illustrated through a number of examples.

Work is under progress to further develop the proposed approach to enable the analysis of uncertainty in the constraints coefficients. Additional work is devoted in the application of the proposed method to scheduling problems.

Acknowledgment

The authors gratefully acknowledge financial support from the National Science Foundation under the NSF CAREER program CTS-9983406.

Literature Cited

1. Roodman GM. Postoptimality analysis in zero-one programming by implicit enumeration. *Naval Res Log Quarterly.* 1972;19:435.
2. Piper CJ, Zoltners AA. Some easy postoptimality analysis for zero-one programming. *Management Sci.* 1976;22:759.
3. Roodman GM. Postoptimality analysis in zero-one programming by implicit enumeration: the mixed integer case. *Naval Res Log Quarterly.* 1974;21:595.
4. Marsten RE, Morin TL. Parametric integer programming: the right-hand side case. *Ann Discr Math.* 1977;1:375.
5. Ohtake Y, Nishida N. A branch-and-bound algorithm for 0-1 parametric mixed integer programming. *Oper Res Lett.* 1985;4:41.
6. Holm S, Klein D. Three methods for post-optimal analysis in integer linear programming. *Math Prog Study.* 1984;21:97.
7. Jenkins L, Peters D. A computational comparison of Gomory and knapsack cuts. *Compt Oper Res.* 1987;14:449.
8. Jenkins L. Parametric methods in integer linear programming. *Ann Oper Res.* 1990;27:77.
9. Crema A. The multiparametric 0-1 integer linear programming problem: a unified approach. *Eur J Oper Res.* 2002;139:511.
10. Acevedo J, Pistikopoulos EN. A multiparametric programming approach for linear process engineering problems under uncertainty. *Ind Eng Chem Res.* 1997;36:717.
11. Pertsinidis A, Grossmann IE, McRae GJ. Parametric optimization of MILP programs and a framework for the parametric optimization of MINLPs. *Comput Chem Eng.* 1998;22:S205.
12. Dua V, Pistikopoulos EN. Algorithms for the solution of multiparametric mixed integer linear optimization problems. *Ann Oper Res.* 2000;99:123.
13. Gal T, Nedoma J. Multiparametric linear programming. *Management Sci.* 1972;18:406.
14. Borrelli F, Bemporad A, Morari M. Geometric approach for multiparametric linear programming. *J Optimization Theory and Applications.* 2003;118:515.
15. Alves MJ, Climaco J. An interactive reference point approach for multiobjective mixed-integer programming using branch-and-bound. *Eur J Oper Res.* 2000;124:478.
16. Bard JF. *Practical bilevel optimization: algorithms and applications.* Kluwer Academic Publishers; 1998.
17. Sahinidis NV. BARON: A general purpose global optimization software package. *J Global Optimization.* 1996;8:201.
18. Ierapetritou MG, Floudas CA. Effective continuous-time formulation for short-term scheduling. 1. Multipurpose batch processes. *Ind Eng Chem Res.* 1998;37:4341.
19. Vin JP, Ierapetritou MG. Robust short-term scheduling of multiproduct batch plants under demand uncertainty. *Ind Eng Chem Res.* 2001; 40:4543.

Manuscript received Aug. 5, 2005, and revision received Nov. 8, 2005, and final revision received Feb. 27, 2006.